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1987 J. Phys. A: Math. Gen. 20 L1077

(http://iopscience.iop.org/0305-4470/20/16/008)

J. Phys. A: Math. Gen. **20** (1987) L1077–L1081. Printed in the UK

LETTER TO THE EDITOR

Exact solutions to anisotropic SU($\mathcal{N} \geqslant 3$) chains†

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Received 26 August 1987

Abstract. Exact eigenfunctions for anisotropic SU($\mathcal{N}$) chains are obtained for the fundamental representations $\mathcal{N}$ and $\bar{\mathcal{N}}$ using a Bethe ansatz. For three anisotropies $\lambda = 0, \pm 1$ eigenfunctions of all possible symmetry types are obtained, otherwise only the completely symmetric and antisymmetric representations are found. It is shown that special and limiting cases of our model include the non-linear Schrödinger model, the spin-$\frac{1}{2}$ *XXZ* model and Uimin's spin-1 exchange Hamiltonian.

Recent progress in the theory of quantum integrable systems have been made possible by the quantum inverse transform method. Starting from an R matrix satisfying the Yang–Baxter equation, families of integrable models connected with given R matrices have been derived. In particular, the *XXZ* model of higher spin (Zamolodchikov and Fateev 1980, Sogo 1984, Kirillov and Reshetikhin 1987) or the multicomponent non-linear Schrödinger model (Pu *et al* 1987) have been studied by this method.

The present letter uses the coordinate Bethe ansatz to solve exactly integrable models based on SU($\mathcal{N}$) symmetry, where $\mathcal{N} = 3, 4, 5, \ldots$, leading to spin models with Schrödinger-type exchange. An anisotropic model with SU($\mathcal{N}$) symmetry and its continuum version have been formulated in an earlier paper (Gölzer and Holz 1987). Below we construct exact solutions to this anisotropic SU($\mathcal{N}$) chain. The exact solutions, which we find by means of a coordinate Bethe ansatz, are either completely symmetric or antisymmetric with respect to particle exchange. For special values of the anisotropy parameter λ, i.e. $\lambda = 0, \pm 1$, all symmetry types are allowed. We also show that special and limiting cases of the anisotropic SU(3) model include a range of well known exactly integrable models of *XXX*, *XXZ* and non-linear Schrödinger type.

Consider a one-dimensional lattice of N sites of the topology of a ring. Let each site be occupied by an object and its internal symmetry space described by an SU($\mathcal{N}$) weight vector. We work within one of the fundamental representations $\mathcal{N}$ or $\bar{\mathcal{N}}$ of SU($\mathcal{N}$). Let the operator $P_{i,j}$ permute whatever objects occupy sites i and j. Then the Hamiltonian for nearest-neighbour exchange (Sutherland 1975, Kulish and Reshetikhin 1981)

$$\mathcal{H} = \varepsilon \sum_{i=1}^{N} P_{i,i+1} \qquad N+1=1, \varepsilon = \pm 1 \tag{1}$$

† Work supported by Deutsche Forschungsgemeinschaft within SFB 130.

0305-4470/87/161077+05\$02.50

can be expressed in terms of the SU($\mathcal{N}$) raising and lowering operators $E\pm\boldsymbol{\alpha}$ and the operators H_0 of the Cartan subalgebra as

$$\mathscr{H}=2\varepsilon\sum_{i=1}^{N}\left(\sum_{p\neq q}E_{\nu^p-\nu^q,i}E_{\nu^q-\nu^p,i+1}+\sum_{p=1}^{\mathcal{N}-1}H_{p,i}H_{p,i+1}\right). \qquad (2)$$

It is also possible to define anisotropic Hamiltonians (Gölzer and Holz 1987), where a is fixed:

$$\mathscr{H}^a=2\varepsilon\sum_{i=1}^{N}\left(\sum_{\substack{q=1\\ q\neq a}}^{\mathcal{N}}(E_{\nu^a-\nu^q,i}E_{\nu^q-\nu^a,i+1}+E_{\nu^a-\nu^q,i+1}E_{\nu^q-\nu^a,i})\right.$$
$$\left.+\lambda\sum_{\substack{p\neq q\\ p\neq a,\,q\neq a}}^{\mathcal{N}}E_{\nu^p-\nu^q,i}E_{\nu^q-\nu^p,i+1}+\lambda\sum_{p=1}^{\mathcal{N}-1}H_{p,i}H_{p,i+1}\right). \qquad (3)$$

We wish to show that eigenfunctions for these operators can be found using a Bethe ansatz. Let the weights be ordered such that ν^1 is the highest weight and $\nu^1>\nu^2>\ldots>\nu^{\mathcal{N}}$. The configuration where each object on the chain is in a state of highest weight is chosen as the vacuum. This picture implies that acting with one of the $\mathcal{N}-1$ operators $E_{-\boldsymbol{\alpha}}\equiv E_{\nu^q-\nu^1}, q=2,\ldots,\mathcal{N}$ on the vacuum creates a particle and there are $\mathcal{N}-1$ different types of particles $\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^{\mathcal{N}-1}$. Hence it is natural to seek the eigenfunctions of the problem in the form

$$|\Psi\rangle=\sum_{j_1<\ldots<j_M}f(j_1,\ldots,j_M\,|\,Q)E_{-\boldsymbol{\alpha},j_1}\ldots E_{-\boldsymbol{\alpha},j_M}|\mathrm{vac}\rangle$$
$$Q\in S_M;\ -\boldsymbol{\alpha}\in\{\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^{\mathcal{N}-1}\}. \qquad (4)$$

This is an M-particle state with M_1 operators of type $E_{\boldsymbol{\alpha}^1}$, M_2 operators of type $E_{\boldsymbol{\alpha}^2}$, etc, acting on the vacuum and creating $M=M_1+M_2+\ldots+M_{\mathcal{N}-1}$ particles. S_M is the permutation group of the M particles.

Note that the amplitude function f does not distinguish which one of the particle species $\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^{\mathcal{N}-1}$ has been created. It is now straightforward to show that (4) is an eigenfunction to (3) with eigenvalue E, if unphysical amplitudes are defined in the form

$$f(\ldots j,j\ldots|\,Q)+f(\ldots j+1,j+1\ldots|\,Q)$$
$$=\lambda[f(\ldots j,j+1\ldots|\,Q)+f(\ldots j+1,j\ldots|\,Q)] \qquad (5)$$

which is equivalent to that used in the Bethe ansatz, and

$$\{\varepsilon E+2\lambda M+\mathrm{constant}\}f(j_1,\ldots,j_M\,|\,Q)=\sum_{k=1}^{M}\sum_{\sigma=\pm1}f(\ldots j_{k+\sigma}\ldots|\,Q). \qquad (6)$$

The solutions of this secular equation are sought in the Bethe ansatz form:

$$f(x_1,\ldots,x_M\,|\,Q)=\sum_{P\in S_M}A(Q,P)\exp\mathrm{i}[p_{P1}x_1+\ldots+p_{PM}x_M] \qquad (7)$$

where P runs over all elements of S_M. Furthermore the replacement $j\to x$ has been made in f, where the ordering $x_1\leqslant x_2\leqslant\ldots\leqslant x_M$ is used for all Q. The energy eigenvalues are then given by the familiar expression

$$E=-2\varepsilon\sum_{j=1}^{M}(\lambda-\cos p_j)+\mathrm{constant}. \qquad (8)$$

Let the $M!$ coefficients $A(Q, P)$, for fixed P, be arranged as a column vector ξ_P. Then it is found that the scattering conditions (5), for example at $x_4 = x_3 + 1$, can be satisfied provided

$$\xi_{\ldots ij\ldots} = Y_{ji}^{34}\xi_{\ldots ji\ldots}$$
$$Y_{ji}^{34} = \frac{x_{ij} + P_{34}}{1 - x_{ij}}. \tag{9}$$

Here P_{34} interchanges $Q3$ and $Q4$, and, for $\lambda \neq 0$, $p_i \neq p_j$:

$$x_{ij} = \frac{1 + e^{i(p_i + p_j)} - \lambda(e^{ip_i} + e^{ip_j})}{\lambda(e^{ip_j} - e^{ip_i})}. \tag{10}$$

Altogether there are $M!(M-1)$ equations of the form (9). They are mutually consistent, if the following identities are satisfied:

$$Y_{ij}^{ab} Y_{ji}^{ab} = 1$$
$$Y_{jk}^{ab} Y_{ik}^{bc} Y_{ij}^{ab} = Y_{ij}^{bc} Y_{ik}^{ab} Y_{jk}^{bc}. \tag{11}$$

In order to discuss the solutions to these identities, it is useful to define new variables:

$$f_j(p_i) = \frac{\lambda}{e^{ip_i} - \lambda} \qquad \tanh l_i = i\tilde{c} f_i \qquad \tilde{c}^2 = \frac{1 - \lambda^2}{\lambda^2}$$

so that

$$x_{ij} = \frac{1 + \tilde{c}^2 f_i f_j}{f_i - f_j} = \frac{i\tilde{c}}{\tanh(l_i - l_j)}. \tag{12}$$

Then it is easy to check that in the following cases the identities (11) are satisfied.

(i) $\lambda = \pm 1$. Equation (12) yields $x_{ij} = (f_i - f_j)^{-1}$ and $x_{ij}^{-1} + x_{jk}^{-1} - x_{ik}^{-1} = 0$. This category of solutions is represented, e.g. by the isotropic Heisenberg ferromagnet or the non-linear Schrödinger model (Yang 1967). There exist no restrictions on the symmetry type of the solution.

(ii) $\lambda = 0$. Equations (9) and (10) yield $Y_{ji}^{ab} = -1$ and this satisfies (11) trivially. This case corresponds to a system of M free particles. There are again no restrictions on the symmetry type of the solution. Accordingly, for our anisotropic model the Bethe ansatz works for $\lambda \pm 1, 0$. If $\lambda \neq \pm 1, 0$, it works only for those representations P_{ab} of S_M for which $P_{ab} = \text{constant}(a, b)$ and $P_{ab}^2 = \text{identity}$ holds:

(iii) antisymmetric representation with $P_{ab} = -1$ and $Y_{ji}^{ab} = -1$ (interactionless case).

(iv) symmetric representation with $P_{ab} = 1$ and $Y_{ji}^{ab} = (1 + x_{ij})/(1 - x_{ij})$ (case of interacting identical bosons).

The condition $P_{ab} = \text{constant}(a, b)$ implies that the amplitudes $A(Q, P)$ do not depend on the specific permutation Q and amplitudes $\tilde{A}(Q, P)$ symmetrised or antisymmetrised with respect to Q must be used for the construction of the eigenfunctions (Bader and Schilling 1984).

(v) Solutions to (11) with no restriction on the symmetry type are found for $\lambda \approx 1$ (weak anisotropy) in the continuum limit $p_i \to ap_i$, where a is the lattice constant ($a \to 0$). Then

$$x_{ij} = \frac{2(1 - \lambda)}{\lambda a} \frac{i}{p_i - p_j}.$$

This yields again a solution of non-linear Schrödinger type with $c_{\mathrm{NLS}} = 2(1 - \lambda)/\lambda a$ or, equivalently, $\lambda = 1 - c_{\mathrm{NLS}} a/2$ ($a \to 0$) (Gölzer and Holz 1987).

As an example of the anisotropic SU($\mathcal{N}$) models we consider the case $\mathcal{N}=3$. Then the Hamiltonian for nearest-neighbour exchange is given by

$$H_1=-2\sum_{i=1}^{N}\left(\sum_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^3}(E^+_{\alpha,i}E^-_{\alpha,i+1}+E^+_{\alpha,i}E^+_{\alpha,i+1})+\lambda(E^+_{\boldsymbol{\alpha}^2,i}E^+_{\boldsymbol{\alpha}^2,i+1}+E^-_{\boldsymbol{\alpha}^2,i}E^+_{\boldsymbol{\alpha}^2,i+1})\right.$$
$$\left.+\lambda(T_{3,i}T_{3,i+1}+T_{8,i}T_{8,i+1}-\tfrac{1}{3})\right). \tag{13}$$

For definiteness, the minus sign in (3) has been chosen, and the constant $\frac{2}{3}\lambda N$ has been added. Here T_3 and T_8 represent the two generators forming the Cartan subalgebra of SU(3), $E^+_{\boldsymbol{\alpha}}$ and $E^-_{\boldsymbol{\alpha}}$ are raising and lowering operators for the weights and the $\boldsymbol{\alpha}$ represent root vectors $\boldsymbol{\alpha}^1=(\frac{1}{2},\sqrt{3}/2)$, $\boldsymbol{\alpha}^2=(1/2,-\sqrt{3}/2)$, $\boldsymbol{\alpha}^3=(1,0)$. Special and limiting cases of this Hamiltonian include the following range of well known exactly integrable models.

(i) The two-component non-linear Schrödinger model with either boson or fermion fields, obtained in the continuum limit (Gölzer and Holz 1987).

(ii) The anisotropic spin-$\frac{1}{2}$ Heisenberg model, obtained when the SU(3) generators T_j are represented by means of spin-$\frac{1}{2}$ matrices S_x, S_y, S_z according to: $T_1=0$, $T_2=\frac{1}{2}S_x$, $T_3=0$, $T_4=0$, $T_5=\frac{1}{2}S_y$, $T_6=0$, $T_7=\frac{1}{2}S_z$, $T_8=0$. This leads immediately to the following representation for H_1:

$$H_1=-\tfrac{1}{2}\sum_{i=1}^{N}(S_{x,i}S_{x,i+1}+S_{y,i}S_{y,i+1}+\lambda S_{z,i}S_{z,i+1})+\tfrac{2}{3}\lambda N. \tag{14}$$

(iii) An anisotropic spin-1 model with tensor interactions, obtained when the SU(3) operators are represented by means of products of spin-1 matrices S_x, S_y, S_z according to:

$$E^+_{\boldsymbol{\alpha}^1}=-2^{-1/2}(S_xS_z)\qquad E^-_{\boldsymbol{\alpha}^1}=-2^{-1/2}(S_zS_x)\qquad E^+_{\boldsymbol{\alpha}^2}=2^{-1/2}(S_xS_y)$$
$$E^-_{\boldsymbol{\alpha}^2}=2^{-1/2}(S_yS_x)\qquad E^+_{\boldsymbol{\alpha}^3}=2^{-1/2}(S_yS_z)\qquad E^-_{\boldsymbol{\alpha}^3}=2^{-1/2}(S_zS_y)$$
$$2T_3=S_y^2-S_z^2\qquad 2T_8=3^{-1/2}(2S_x^2-S_y^2-S_z^2).$$

This leads to the following representation for H_1:

$$H_1=-\sum_{i=1}^{N}\left[(\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1})^2+(\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1})+(\lambda-1)\sum_{\mu=1}^{3}(S_\mu S_\mu)_i(S_\mu S_\mu)_{i+1}+\tfrac{1}{2}(\lambda-1)S_i^zS_{i+1}^z\right]$$
$$+2\lambda N. \tag{15}$$

Isotropic exchange of this type has been studied earlier (Uimin 1970) and so we recover for $\lambda=1$ the spin-1 model with Schrödinger-type exchange interaction (Schrödinger 1941) solved by Uimin by means of a Bethe ansatz, namely

$$H_1=-\sum_{i=1}^{N}[(\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1})^2+(\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1})-2]. \tag{16}$$

Using the following identity

$$(\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1})^2=\tfrac{1}{4}T_i^{\mu\nu}T_{i+1}^{\mu\nu}-\tfrac{1}{2}\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1}\qquad T_i^{\mu\nu}\equiv S_i^\mu S_i^\nu+S_i^\nu S_i^\mu \tag{17}$$

the last expression for H_1 can be rewritten as:

$$H_1=-\tfrac{1}{2}\sum_{i=1}^{N}[(\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1})+\tfrac{1}{2}T_i^{\mu\nu}T_{i+1}^{\mu\nu}-4] \tag{18}$$

which is a spin-1 model solved independently by Sutherland (1975) also by means of a Bethe ansatz. Finally we point out that similar representations can be found for the SU($\mathcal{N}>3$) models. With respect to the general solution of the anisotropic SU($\mathcal{N}$) models we remark that the ansatz (7) based on trigonometric functions does not work.

References

Bader H P and Schilling R 1984 *Phys. Rev.* B **30** 3770
Gölzer B and Holz A 1987 *J. Phys. A: Math. Gen.* **20** 3327
Kirillov A N and Reshetikhin N Yu 1987 *J. Phys. A: Math. Gen.* **20** 1565, 1587
Kulish P P and Reshetikhin N Yu 1981 *Sov. Phys.-JETP* **53** 108
Ogievetsky E and Wiegmann P 1986 *Phys. Lett.* **168B** 360
Pu F C, Wu Y Z and Zhao B H 1987 *J. Phys. A: Math. Gen.* **20** 1173
Schrödinger E 1941 *Proc. R. Irish Acad.* A **47** 39
Sogo K 1984 *Phys. Lett.* **104A** 51
Sutherland B 1975 *Phys. Rev.* B **12** 3795
Uimin G V 1970 *Zh. Eksp. Teor. Fiz. Pis. Red.* **12** 332 (1970 *JETP Lett.* **12** 225)
Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
Zamolodchikov A B and Fateev V A 1980 *Sov. J. Nucl. Phys.* **32** 298